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Asymptotic Ruin Probabilities of an Entrance Processes Based Risk Model with Interest Force and Regularly Varying Claims*

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Abstract: Investigated in this paper is an insurance risk model with a constant interest force and the claim process being driven by an entrance process. Under the conditions that the entrance process is a renewal process and the claim size is of regularly varying tailed, the asymptotic behavior for the ruin probability as the initial capital tends to infinity is obtained. A similar result also holds for the case that the entrance process is a homogeneous Poisson process.

Keywords: asymptotic; regular variation; heavy tailed; ruin probability; renewal process; poisson process

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1 Introduction

Asymptotic behavior of the ruin probabilities for the insurance risk model is a hot topic in the insurance and actuarial science. In the classical non-life insurance risk model, i.e. the Cramér-Lundberg model, the surplus process is stated as

$$U(t) = u + ct - S(t), \tag{1}$$

where $u \geq 0$ is the initial capital of the insurance company, $c > 0$ is the premium income rate and

$$S(t) = \sum_{j=1}^{M(t)} Y_j$$

is the total claim amount, in which $M(t)$ is the claim number process which is modelled by a homogeneous Poisson process and Y_j is the j-th claim size satisfying that Y_j, $j = 1, 2, \cdots$, are i.i.d. random variables.

The ruin probability for the model (1) is defined as

$$\psi_U(u) = \mathbf{P}\Big(\inf_{0\leq t<\infty} U(t) < 0 \,\Big|\, U(0) = u\Big). \tag{2}$$

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As the initial capital u tends to infinity, the asymptotic behavior of the ruin probability $\psi_U(u)$ has been investigated by Cramér and Lundberg in 1930's, see, for example, [1].

With the development of science and technology, the classical insurance risk model has more and more extensions, one of which is about the extension of the claim size. The claim size can be generalized to a general non-negative random variable. Among the class of such kind of random variables, the heavy-tailed distributed variables is a typical one. For the case that the claim size is of subexponential, under certain conditions, the asymptotic behavior of the ruin probability $\psi_U(u)$ as $u \to \infty$ has been investigated by Embrechts and Veraverbeke ([2]).

Another extension of the classical Cramér-Lundberg model is that the insurance company could invest its money into the riskless bond and receives interest on its reserve. Suppose that the constant force of the interest is $\delta > 0$. Then the surplus process can be written as

$$U^\delta(t) = u\mathrm{e}^{\delta t} + c\int_0^t \mathrm{e}^{\delta s}\mathrm{d}s - \int_0^t \mathrm{e}^{\delta(t-v)}\mathrm{d}S(v). \tag{3}$$

Apparently, the classical model is a special case of $\delta = 0$ in this formula. The ruin probability $\psi_U^\delta(u)$ for the model (3) is defined similarly to that of $\psi_U(u)$. Asymptotic results for the ruin probabilities $\psi_U^\delta(u)$ under the condition that the claim size is of light-tailed has been investigated by Sundt and Teugels ([3]). For the case that the claim size is a regularly varying tailed random variable, which belongs to a subclass of the heavy-tailed random variables, similar behavior has been studied by Klüppelberg and Stadtmüller ([4], see also [5-7]) and these results have been extended to the case of renewal processes in [8].

In the classical risk model and all its extensions, the premium processes are looked at from a macro point of view. As a matter of fact, a practical point is to assume that the claim number process should be driven by the entrance process, which is actually the arriving process of the insurants. Based on this idea, a new non-life insurance risk model has been presented in [9], which can be regarded as a martingale perturbed version of the generalization of the Cramér-Lundberg risk model. Some limiting results for the new risk model have been obtained by the authors.

Given the insurance risk model in [9], a natural and interesting question is to investigate its ruin probability. Such a work has been done in [10], in which the author considered a particular entrance processes based risk model and gave the infinite-time ruin probability under the condition that the claim size is of light-tailed. For the case that the claim size is of subexponential, a simply asymptotic behavior for the finite-time ruin probability is established recently in [11]. In this paper, we consider the infinite-time ruin probability under the condition that the distribution function of the claim size is of regularly varying tailed and give an asymptotic behavior of the ruin probability. Our assumptions on the claims are applied, for instance, to Pareto, loggamma, certain Benktander and stable claim size distributions. The paper is organized as the following, Section 2 gives the description of the model. The main results and its proofs will be presented in Section 3.

2 Model description

Let $u > 0$ be the initial capital of an insurance company, and S_j be the arrival point of the

j-th insurant and $0 < S_1 < S_2 < \cdots$. The associated counting process

$$N_0(t) = \sum_{j=1}^{\infty} \mathbf{1}_{\{S_j \le t\}}$$

counts the number of arrival insurants up to time t, which is intepreted as the entrance process.

We suppose that the insurance company provides K different policies, the validity times of which are $0 < a_1 < a_2 < \cdots < a_K$. Upon the arrival, the j-th insurant will choose a policy with the validity time C_j with probability $\mathbf{P}(C_j = a_\ell) = p_\ell$, $\ell = 1, 2, \cdots, K$. The insurer charges a premium $f(C_j)$ for the j-th policy according to its validity time. $f(\cdot)$ is supposed to be a strictly increasing positive deterministic function.

Let T_j be the time period from the epoch at which the j-th customer buys a policy to the epoch he/she suffers a loss insured against it. T_j, $j = 1, 2, \cdots$ are supposed to be i.i.d. with the common distribution function $G(\cdot)$. It is easy to see that the following point process is the number of claims up to time t,

$$N(t) = \sum_{j=1}^{N_0(t)} \mathbf{1}_{\{S_j + T_j \wedge C_j \le t, T_j \le C_j\}}, \tag{4}$$

where $a \wedge b = \min\{a, b\}$. Let Y_j be the j-th possible claim size and suppose that Y_j, $j = 1, 2, \cdots$ are i.i.d. with the common distribution function $F(\cdot)$. Under the assumption that the insurer incurring more than one claims from one policy is small enough to be neglected, then the surplus process of the company can be written as the following[9]

$$R(t) = u + \sum_{j=1}^{N_0(t)} f(C_j) - \sum_{j=1}^{N_0(t)} Y_j \mathbf{1}_{\{S_j + T_j \wedge C_j \le t, T_j \le C_j\}}, \quad t \ge 0,$$

where $\{C_j, j \ge 1\}$, $\{T_j, j \ge 1\}$, $\{Y_j, j \ge 1\}$ and $\{S_j, j \ge 1\}$ are supposed to be independent families of random variables. Note that for the convenience in the following, we suppose that random variables C, T and Y have the same distributions as the common ones of $\{C_j, j \ge 1\}$, $\{T_j, j \ge 1\}$ and $\{Y_j, j \ge 1\}$, respectively.

Suppose that the insurance company invests all its assets into bank and receives interest on its reserve. We assume that the interest rate is a constant $\delta > 0$, then the total surplus of the company up to time t, denoted by $R^\delta(t)$, can be written as

$$R^\delta(t) = u\mathrm{e}^{\delta t} + \sum_{j=1}^{N_0(t)} f(C_j)\mathrm{e}^{\delta(t - S_j)} - \sum_{j=1}^{N_0(t)} Y_j \mathrm{e}^{\delta(t - S_j - T_j)} \mathbf{1}_{\{S_j + T_j \wedge C_j \le t, T_j \le C_j\}}.$$

The definition of the ruin probability $\psi_R^\delta(u)$ is similar to that in (2). The purpose of this paper is to investigate the asymptotic behavior of the ruin probability $\psi_R^\delta(u)$ as u tends to infinity under the condition that the claim size is of regularly varying tailed distributed.

If $F(x)$ is a distribution function, then we say that $\bar{F}(x) = 1 - F(x)$ is a regularly varying with index $\alpha \in \mathbb{R}$, denoted by $\bar{F} \in \mathcal{R}_\alpha$, if

$$\lim_{x \to \infty} \frac{\bar{F}(xt)}{\bar{F}(x)} = t^\alpha, \quad \forall\, t > 0.$$

It implies from [4] that if $\bar{F} \in \mathcal{R}_\alpha$, then there is some slowly varying function $L(\cdot)$ such that

$$\bar{F}(x) = x^\alpha L(x). \tag{5}$$

It is evidently that α in formula (5) should be negative. For the simplicity in the following, if H is the distribution function of a random variable X and $\bar{H} \in \mathcal{R}_\alpha$, then we also write $X \in \mathcal{R}_\alpha$ if it doesn't make any ambiguity. Note that a random variable with regularly varying tailed is of heavy-tailed. As a subclass of subexponential random variables, the class of regularly varying function includes the following distribution functions as its special cases: Pareto, loggamma, certain Benktander and stable claim size distribution (see, e.g., [4] and [12] for details).

Before ending this section, let's give a proposition on the selected point process $\{N(t),\ t \geq 0\}$ defined in (4) under the condition that the entrance process is a Poisson process.

Proposition 1 Suppose that $\{N_0(t), t \geq 0\}$ is a homogeneous Poisson process with intensity $\lambda > 0$, then the selected claim number process $\{N(t), t \geq 0\}$ is a non-homogeneous Poisson process with the intensity function given by

$$\lambda(t) = \lambda G(t) \sum_{\ell=1}^{K} p_\ell G(a_\ell).$$

Proof From the queueing theory point of view, $\{N(t), t \geq 0\}$ is actually the departure process of an $M/G/\infty$ queueing system. Since each customer is selected with probability

$$\mathbf{P}(T \leq C) = \sum_{\ell=1}^{K} p_\ell G(a_\ell),$$

it implies from the properties of Poisson process that the practical arrival process is a Poisson process with the intensity given by

$$\lambda \sum_{\ell=1}^{K} p_\ell G(a_\ell).$$

The conclusion of example 2-5-1 in [13] gives the result of this proposition. This completes the proof.

3 Main results and its proofs

Here is the main result.

Theorem 1 Let $\{N_0(t), t \geq 0\}$ be a renewal process, the i.i.d. interarrival time of which is $\{D_i, i \geq 1\}$ with the commonly non-negative distributed random variable being denoted by D. In addition, let the claim size Y is of regularly varying tailed, i.e. $\bar{F} \in \mathcal{R}_\alpha$ for some $\alpha < 0$, then we have that

$$\psi_R^\delta(u) \sim \frac{\beta \mathbf{E}\mathrm{e}^{\delta\alpha D}}{1 - \mathbf{E}\mathrm{e}^{\delta\alpha D}} \bar{F}(u),$$

where

$$\beta = \sum_{\ell=1}^{K} p_\ell \int_0^{a_\ell} \mathrm{e}^{\alpha\delta s} \mathrm{d}G(s)$$

is a constant.

Note that if we define the discounted value of $R^\delta(t)$ as the following

$$\begin{aligned} V^\delta(t) &= R^\delta(t)\mathrm{e}^{-\delta t} \\ &= u + \sum_{j=1}^{N_0(t)} \Big(f(C_j)\mathrm{e}^{-\delta S_j} - Y_j \mathrm{e}^{-\delta(S_j+T_j)} \mathbf{1}_{\{S_j+T_j \wedge C_j \le t, T_j \le C_j\}} \Big), \end{aligned}$$

then it is easy to see that $\psi_R^\delta(u) = \psi_V^\delta(u)$ for each $u \ge 0$. We will use this process and the proving method of Theorem 1 in [8] to investigate the asymptotic behavior of the ruin probability. Before proving the main theorem, let's give a lemma which will play a key role in the procedure of the proof.

Lemma 1 Let the claim size Y be a random variable of regularly varying tailed with index $\alpha < 0$, then $Y\mathbf{1}_{\{T\le C\}}\mathrm{e}^{-\delta T} \in \mathcal{R}_\alpha$, moreover,

$$\mathbf{P}\big(Y\mathbf{1}_{\{T\le C\}}\mathrm{e}^{-\delta T} > x\big) \sim \beta\,\mathbf{P}(Y > x), \qquad \text{as} \qquad x \to \infty,$$

where β is the same constant as that in Theorem 1.

Proof Since $\bar{F} \in \mathcal{R}_\alpha$, there is a slowly varying function $L(\cdot)$ such that $\bar{F}(x) = x^{-\alpha}L(x)$, $x > 0$. It implies from the case (b) of Theorem A3.2 in [12] that

$$\lim_{x\to\infty} \frac{L(xt)}{L(x)} = 1$$

uniformly in t on the interval $[1, \mathrm{e}^{\delta a_K}]$, therefore, from the dominated convergence theorem we have that

$$\begin{aligned} \mathbf{P}\big(Y\mathbf{1}_{\{T\le C\}}\mathrm{e}^{-\delta T} > x\big) &= \sum_{\ell=1}^{K} p_\ell \int_0^{a_\ell} \mathbf{P}(Y > x\mathrm{e}^{\delta s})\mathrm{d}G(s) \\ &= \sum_{\ell=1}^{K} p_\ell \int_0^{a_\ell} (x\mathrm{e}^{\delta s})^\alpha L(x\mathrm{e}^{\delta s})\mathrm{d}G(s) \\ &\sim x^\alpha L(x) \sum_{\ell=1}^{K} p_\ell \int_0^{a_\ell} \mathrm{e}^{\alpha\delta s}\mathrm{d}G(s) = \beta x^\alpha L(x). \end{aligned}$$

The proof is completed.

Proof of Theorem 1 From the definition of $V^\delta(u)$, it is easy to see that

$$V^\delta(t) \ge u - \sum_{j=1}^{\infty} \mathrm{e}^{-(T_j+S_j)\delta} Y_j I_{\{T_j \le C_j\}}.$$

Recall the properties of renewal processes, using Lemma 1 in [8], Lemma 1 and the above

inequality, we have

$$\begin{aligned}
\psi_V^\delta(u) &\le \mathbf{P}\Big(\sum_{j=1}^{\infty} \mathrm{e}^{-\delta S_j}\big(Y_j \mathbf{1}_{\{T_j \le C_j\}} \mathrm{e}^{-\delta T_j}\big) > u\Big) \\
&\sim \mathbf{P}\big(Y \mathbf{1}_{\{T \le C\}} \mathrm{e}^{-\delta T} > u\big) \sum_{n=1}^{\infty} \mathbf{E}(\mathrm{e}^{-\delta S_n})^{-\alpha} \\
&= \beta \bar{F}(u) \sum_{n=1}^{\infty} \mathbf{E}\mathrm{e}^{\alpha\delta(D_1+D_2+\cdots+D_n)} \\
&= \beta \bar{F}(u) \sum_{n=1}^{\infty} (\mathbf{E}\mathrm{e}^{\alpha\delta D})^n = \frac{\beta \mathbf{E}\mathrm{e}^{\delta\alpha D}}{1-\mathbf{E}\mathrm{e}^{\delta\alpha D}} \bar{F}(u).
\end{aligned}$$

The definition of $V^\delta(u)$ also implies that

$$V^\delta(u) \le u + \sum_{j=1}^{\infty} f(C_j)\mathrm{e}^{-S_j\delta} - \sum_{j=1}^{N_0(t)} \mathrm{e}^{-(T_j+S_j)\delta} Y_j I_{\{S_j+T_j\wedge C_j \le t, T_j \le C_j\}},$$

from which we have that

$$\begin{aligned}
\psi_V^\delta(u) &\ge \mathbf{P}\Big(\sum_{j=1}^{N_0(t)} \mathrm{e}^{-(T_j+S_j)\delta} Y_j \mathbf{1}_{\{S_j+T_j\wedge C_j \le t, T_j \le C_j\}} - \sum_{j=1}^{\infty} f(C_j)\mathrm{e}^{-\delta S_j} > u \ \text{ for some } \ t \ge 0\Big) \\
&= \mathbf{P}\Big(\sum_{j=1}^{\infty} \mathrm{e}^{-\delta S_j}\big(Y_j \mathrm{e}^{-\delta T_j} \mathbf{1}_{\{T_j \le C_j\}} - f(C_j)\big) > u\Big).
\end{aligned}$$

To complete the proof of the theorem, it is sufficient to prove that

$$\mathbf{P}\big(Y\mathrm{e}^{-\delta T}\mathbf{1}_{\{T \le C\}} - f(C) > u\big) \sim \mathbf{P}\big(Y\mathbf{1}_{\{T \le C\}}\mathrm{e}^{-\delta T} > u\big),$$

which can be obtained from the fact that $Y\mathbf{1}_{\{T \le C\}}\mathrm{e}^{-\delta T} \in \mathcal{R}_\alpha \subset \mathcal{L}$, the class of random variables with long tailed distributions (see [12] for details), and $f(a_1) \le f(C) \le f(a_K)$. This completes the proof.

It implies from Theorem 1 that we have the following corollary.

Corollary 1 Under the conditions in Theorem 1, if the entrance process $\{N_0(t), t \ge 0\}$ is a homogeneous Poisson process with intensity $\lambda > 0$, then we have that

$$\psi_R^\delta(u) \sim \frac{\lambda\beta}{(-\alpha)\delta} \bar{F}(u),$$

where β is the same constant as defined in Theorem 1.

Proof If $\{N_0(t), t \ge 0\}$ is a homogeneous Poisson process with intensity $\lambda > 0$, then the random variable D, the interarrival time of the renewal process, is of exponential distributed with the parameter given by λ. Straightforward calculation indicates that

$$\mathbf{E}\mathrm{e}^{\delta\alpha D} = \frac{\lambda}{\lambda - \delta\alpha},$$

which completes the proof.

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基于进入过程具有常利率和正则变化索赔的风险模型的渐近破产概率

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摘　要: 本文研究了一类带常利率的，并且索赔过程由进入过程驱动的风险保险模型。在进入过程是一般更新过程以及索赔额是正则尾分布的条件下，得到了当初始资本趋于无穷时，破产概率的渐近行为，类似的结论对于进入过程是齐次泊松过程的情形也同样成立。

关键词: 渐近性；正则变化；重尾；破产概率；更新过程；泊松过程